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Whispering-gallery modes and resonances of an elliptic cavity

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Abstract

Eigenfunctions of the whispering-gallery type in elliptic cavities are considered. Asymptotic expansions for resonances are derived from the uniform asymptotic expansions of Mathieu functions and modified Mathieu functions constructed by applying the Langer–Olver method. These asymptotic expansions are improved by including exponentially small terms which lie beyond all orders of the perturbative series and can be captured by carefully taking into account Stokes's phenomenon. A classification of resonances along the four irreducible representations of $\mathcal{C}_{2v}$ (the symmetry group of the elliptic cavity) is provided, and the splitting up of resonances is then understood in connection with the breaking of O(2)-symmetry (invariance under any rotation).

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1. Introduction

Whispering-gallery phenomena are connected with the propagation of waves within closed convex regions and have been well known since their observation, at the inside surface of the dome of St Paul's Cathedral in London, by Herschel and Airy. They were described by Rayleigh qualitatively in 1887 [1] and understood by him quantitatively in 1910 [2]. In a convex cavity, whispering-gallery modes are very particular resonant modes that exist only for certain values of the resonance frequency. They are concentrated in the neighborhood of the boundary of the cavity, in a layer whose thickness is proportional to $k^{-2/3}$ with k the wavenumber, and are exponentially damped outside this layer [3].

In this paper, we consider the whispering-gallery modes of an elliptic cavity (i.e. an infinite cylinder of elliptic cross section) and the associated resonances. We are mainly concerned with their high-frequency asymptotic behaviour ($k \to \infty$). We only consider the Dirichlet boundary condition on the surface of the cavity. Furthermore, we assume that the problem is invariant under any translation along the cylinder axis and thus reduces to a two-dimensional problem. Such a problem can then be encountered (i) in electromagnetism, in the study of microwaves in a metallic elliptic cavity, (ii) in acoustics, in the study of ultrasonic waves in a soft elliptic

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cavity or in the study of the transverse vibrations of a stretched elliptic membrane held at its boundary, and (iii) in quantum mechanics, in the study of a quantum particle in an elliptic billiard.

In a circular cavity, the determination of whispering-gallery modes and of the associated resonances is an easy task because the Helmholtz equation can be solved exactly by separation of variables. From their high-frequency asymptotic behaviours, it is possible to guess and then to construct the asymptotic expansions for the whispering-gallery modes of an arbitrary convex cavity and for the corresponding resonances (see chapter 7 of [3]). Such expansions are very useful and could be used in the context of the elliptic cavity. Unfortunately, because of their purely perturbative nature, the asymptotic expansions for resonances cannot take into account subtle effects associated with corrective terms lying beyond all orders of the perturbative series. In the context of the elliptic cavity, they do not include the non-perturbative corrections (exponentially small terms) corresponding to the spitting of resonances which occurs in the transition from the circular to the elliptic cavity and which can be explained in terms of the symmetry breaking $O(2) \to \mathcal{C}_{2v}$.

In order to derive the high-frequency asymptotic behaviour of the whispering-gallery modes and of the associated resonances and to quantitatively understand their splitting, we prefer a method which presents analogies with the approach previously developed by us in our study of scattering by an elliptic cylinder [4] and which permits us to directly use some of the results obtained in that context. We construct the resonant modes of the elliptic cavity in terms of Mathieu and modified Mathieu functions and we classify them along the four irreducible representations A_1, A_2, B_1 and B_2 of the symmetry group $\mathcal{C}_{2v}$ of that cavity [5]. This is an easy task in this geometry because the Helmholtz equation is separable in elliptic coordinates. Rather than follow the traditional method of analysis [6], and adopt periodic boundary conditions, we assume that the angular coordinate lies in the interval $]-\infty, +\infty[$ and therefore we solve the Helmholtz equation in a region of an infinitely sheeted Riemann surface. The determination of the radial part of the resonant modes (modified Mathieu functions) is then an eigenvalue problem, which is analogous to the determination of the Regge poles in the scattering problem. When we impose periodicity conditions on the angular part of the modes (Mathieu functions), we obtain four transcendental equations for the resonances, each one associated with an irreducible representation of $\mathcal{C}_{2v}$. In order to solve these equations, we consider the uniform asymptotic expansions of Mathieu and modified Mathieu functions constructed (in terms of Airy functions) by applying the Langer–Olver method [7, 8]. We then perturbatively solve the equations for the resonances and, by carefully taking into account Stokes's phenomenon, we exponentially improve the asymptotic expansion for resonances.

It should be noted that the present paper is linked with previous work done in the context of the semiclassical and exact quantizations of the elliptic billiard, a topic which has been the subject of a rather large number of recent investigations [9–17]. The splitting up of resonances (or more exactly of energy levels) of the elliptic billiard has been considered in the context of Einstein–Brillouin–Keller (EBK) quantization [18, 19] and described by using uniform EBK quantization rules (see e.g. [9, 10, 14] or, for a slightly different approach, [15]). In these approaches, the whole spectrum of the elliptic billiard is described, while asymptotic formulae for the energy levels are not explicitly obtained. Our method is different. It is only valid for resonances associated with whispering-gallery modes but it provides asymptotic formulas for resonances. We believe that it could be generalized for the whole spectrum by using uniform asymptotic expansions of Mathieu and modified Mathieu functions constructed in terms of Weber functions.

All the numerical calculations and some algebraic ones have been performed with *Mathematica* [20]. The numerical results displayed in the paper have been obtained using this program. The main advantage with *Mathematica* is that it can be configured to work out with any specified accuracy. 'Exact' results are displayed to six decimal places but are known to 20 decimal places. Such a precision is sometimes required to properly extract the splitting. Asymptotic results have been obtained by truncating asymptotic expansions near their least term.

2. The example of the circular cavity

2.1. Whispering-gallery modes of a circular cavity

Let us first consider a circular cavity, i.e. a region Ω bounded by a circle of radius a in the plane Oxy. We shall use the polar coordinate system (ρ, φ), defined with respect to the centre O of the circle. The resonant modes $\Psi(\rho, \varphi)$ of the cavity Ω and the associated resonances k are the solutions of the Helmholtz equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Psi}{\partial\varphi^2}+k^2\Psi=0 \tag{1}$$

subject to the Dirichlet boundary condition $\Psi(\rho = a, \varphi) = 0$. The method of separation of variables provides two distinct families of solutions in the form

$$\Psi_\nu^{(c)}(\rho,\varphi) = J_\nu(k\rho)\cos\nu\varphi \tag{2a}$$

$$\Psi_\nu^{(s)}(\rho,\varphi) = J_\nu(k\rho)\sin\nu\varphi \tag{2b}$$

where ν is a separation constant and where $J_\nu(x)$ denotes the Bessel function of the first kind [21]. Equations (2a) and (2a) are respectively even and odd in the transformation $\varphi \to -\varphi$. We first assume that $\varphi \in]-\infty, +\infty[$. Ω is then a region of an infinitely sheeted Riemann surface [3]. The Dirichlet boundary condition reads

$$J_\nu(ka) = 0 \qquad \forall\, ka. \tag{3}$$

We first determine its solutions, i.e. the eigenvalues $\nu = \nu_\ell(ka)$ (with $\ell \in \mathbb{N}^*$). Resonances $(ka)_{n,\ell}$ are then sought as the particular values of the reduced wavenumber ka providing periodic solutions of period 2π, i.e. as the solutions of the equation

$$\nu_\ell(ka) = n \tag{4}$$

with $n \in \mathbb{N}$. Whispering-gallery modes are associated with the resonances $(ka)_{n,\ell}$ closest to the corresponding n. These particular resonances arise for $\ell = 1$ and sufficiently large n. Thus the eigenfunctions of whispering-gallery type are given by

$$\Psi_n^{(c)}(\rho,\varphi) = J_n\left((ka)_{n,1}\frac{\rho}{a}\right)\cos n\varphi \tag{5a}$$

$$\Psi_n^{(s)}(\rho,\varphi) = J_n\left((ka)_{n,1}\frac{\rho}{a}\right)\sin n\varphi \tag{5b}$$

and a given resonance $(ka)_{n,1}$ is twofold degenerate.

2.2. Asymptotic expansions for the eigenvalues $\nu_\ell(ka)$

We now determine the asymptotic expansions (for $ka \to +\infty$) for the eigenvalues $\nu_\ell(ka)$, by solving (3). With this aim in view, we need the uniform asymptotic expansion for the Bessel function of the first kind [21, 22] obtained by using the Langer–Olver method [7, 8] and valid for $|\nu| \to \infty$ and $k\rho \simeq \nu$,

$$J_\nu(k\rho) = \sqrt{2}\left(\frac{\nu^2\zeta}{\nu^2 - k^2\rho^2}\right)^{1/4}\left\{\frac{\mathrm{Ai}(\nu^{2/3}\zeta)}{\nu^{1/3}}\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{\nu^{2s}} + \frac{\mathrm{Ai}'(\nu^{2/3}\zeta)}{\nu^{5/3}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{\nu^{2s}}\right\}. \tag{6}$$

Here, ζ is given by

$$\frac{2}{3}\zeta^{3/2} = -\int_1^{k\rho/\nu} \left(\frac{1-z^2}{z^2}\right)^{1/2} \mathrm{d}z \tag{7}$$

and is chosen so that it is real when ν is real and positive, and $k\rho/\nu \in [0, 1]$. Furthermore, $\mathrm{Ai}(x)$ denotes the Airy function, which plays a big role throughout the paper. For more precisions about this function, we refer to chapter 10 of [21]. The coefficients $A_s(\zeta)$ and $B_s(\zeta)$ are defined by recurrence relations and are given explicitly in [21, 22]. We only consider the first two leading terms of (6) and we use

$$A_0(\zeta) = 1 \tag{8a}$$

$$B_0(\zeta) = -\frac{5}{48\zeta^2} + \frac{1}{24\zeta^{1/2}}\left[\frac{5\nu^3}{(\nu^2 - k^2\rho^2)^{3/2}} - \frac{3\nu}{(\nu^2 - k^2\rho^2)^{1/2}}\right]. \tag{8b}$$

To this degree of approximation,

$$J_\nu(ka) = \sqrt{2}\left(\frac{\nu^2\zeta}{\nu^2 - k^2a^2}\right)^{1/4}\left\{\frac{\mathrm{Ai}(\nu^{2/3}\zeta)}{\nu^{1/3}} + B_0(\zeta)\frac{\mathrm{Ai}'(\nu^{2/3}\zeta)}{\nu^{5/3}}\right\}\left[1 + \underset{|\nu|\to\infty}{\mathrm{O}}\left(\frac{1}{\nu^2}\right)\right] \tag{9}$$

and the condition $J_\nu(ka) = 0$ can be written

$$\frac{\mathrm{Ai}(\nu^{2/3}\zeta)}{\mathrm{Ai}'(\nu^{2/3}\zeta)} \approx -\nu^{-4/3}B_0(\zeta). \tag{10}$$

The eigenvalues $\nu_\ell(ka)$ can be sought around the solutions of

$$\mathrm{Ai}(\nu^{2/3}\zeta) = 0 \tag{11}$$

i.e. as the solutions of

$$\nu\zeta^{3/2} = (x_\ell + \delta x)^{3/2} \tag{12}$$

where x_ℓ ($\ell \in \mathbb{N}^*$) is the lth zero of the Airy function $\mathrm{Ai}(x)$ (see table 1 for the first five numerical values). By expanding in (10) $\mathrm{Ai}(x)$ and $\mathrm{Ai}'(x)$ in Taylor series about x_ℓ, we obtain

$$\delta x = -\nu^{-4/3}B_0(\zeta). \tag{13}$$

Equation (12) can be solved step by step and we finally obtain, for the eigenvalues, the following asymptotic expansions:

$$\begin{aligned}\nu_\ell(ka) = {} & ka + x_\ell\left(\frac{ka}{2}\right)^{1/3} + \frac{x_\ell^2}{60}\left(\frac{ka}{2}\right)^{-1/3} - \frac{x_\ell^3 + 10}{1400}\left(\frac{ka}{2}\right)^{-1} \\ & + \frac{281\,x_\ell^4 + 10\,440\,x_\ell}{4536\,000}\left(\frac{ka}{2}\right)^{-5/3} - \frac{73\,769\,x_\ell^5 + 6624\,900\,x_\ell^2}{10\,478\,160\,000}\left(\frac{ka}{2}\right)^{-7/3} \\ & + \frac{93\,617\,x_\ell^6 + 16\,495\,400\,x_\ell^3 - 1744\,600}{100\,900\,800\,000}\left(\frac{ka}{2}\right)^{-3} + \underset{ka\to+\infty}{\mathrm{O}}[(ka)^{-11/3}].\end{aligned} \tag{14}$$

Up to this order of the asymptotic expansion (14), the terms corresponding to $s = 0$ into (6) are sufficient.

It should be noted that the present eigenvalue problem is analogous to the determination of the Regge poles in the scattering problem considered in [4]. In particular, the expression (14) can be recovered from the asymptotic expansions for Regge poles (see [4, equation (25)]) by changing x_ℓ in $\mathrm{e}^{2\mathrm{i}\pi/3}x_\ell$.

Table 1. Zeros x_ℓ of the Airy function Ai(x).

ℓ	x_ℓ
1	−2.338 107 410 459 774
2	−4.087 949 443 057 359
3	−5.520 559 828 095 544
4	−6.786 708 090 071 631
5	−7.944 133 587 393 717

2.3. Asymptotic expansions for resonances

By taking for $\nu_\ell(ka)$ its asymptotic expansion (14) and then by inverting equation (4), we obtain

$$(ka)_{n,\ell} = n - x_\ell \left(\frac{n}{2}\right)^{1/3} + \frac{3\,x_\ell^2}{20}\left(\frac{n}{2}\right)^{-1/3} + \frac{x_\ell^3+10}{1400}\left(\frac{n}{2}\right)^{-1} - \frac{479\,x_\ell{}^4 - 40\,x_\ell}{504\,000}\left(\frac{n}{2}\right)^{-5/3}$$
$$-\frac{20\,231\,x_\ell{}^5 + 55\,100\,x_\ell{}^2}{129\,360\,000}\left(\frac{n}{2}\right)^{-7/3} + \underset{n\to+\infty}{\mathrm{O}}(n^{-3}). \qquad (15)$$

As previously noted, whispering-gallery modes correspond to the resonances $(ka)_{n,\ell}$ closest to the corresponding n. These particular resonances arise for $\ell = 1$ and sufficiently large n. This is consistent with the domain of validity of the uniform asymptotic expansion (6) ($|\nu| \to \infty$ and $ka \simeq \nu$ with $\nu \simeq n$) considered in the calculations.

3. The elliptic cavity

3.1. Geometry and symmetry considerations

Let us now consider an elliptic cavity, i.e. a region Ω bounded by an ellipse in the plane Oxy. We introduce the elliptic coordinates (ξ, η), related to the rectangular ones (x, y) by the transformation

$$x = c\cosh\xi\,\cos\eta \qquad y = c\sinh\xi\,\sin\eta \qquad (16)$$

where $0 \leqslant \xi < \infty$ and $-\pi \leqslant \eta \leqslant \pi$. The equation $\xi = \xi_0$ defines the surface of an ellipse whose eccentricity is $e = 1/\cosh\xi_0$ and whose foci are located at $(x = \pm c, y = 0)$. The limiting case $\xi_0 \to +\infty$ corresponds to the circle.

The transition from the circular cavity to the elliptic one corresponds to the breaking of O(2) symmetry (invariance under any rotation about the Oz axis, perpendicular to the plane Oxy). However, the elliptic cavity remains invariant under four symmetry transformations: E, the identity transformation ($\eta \to \eta$); C_2, the rotation through π about the Oz axis ($\eta \to \pi+\eta$); π_x, the mirror reflection in the plane Oxz ($\eta \to -\eta$) and π_y, the mirror reflection in the plane Oyz ($\eta \to \pi-\eta$). These four transformations form the finite group $\mathcal{C}_{2\mathrm{v}}$. Four one-dimensional irreducible representations labelled A_1, A_2, B_1 and B_2 are associated with the symmetry group $\mathcal{C}_{2\mathrm{v}}$ [5].

Consequently, any function V of the 'angular' coordinate η can be expanded in terms of these four irreducible representations as

$$V(\eta) = V^{(A_1)}(\eta) + V^{(A_2)}(\eta) + V^{(B_1)}(\eta) + V^{(B_2)}(\eta) \qquad (17)$$

with the components $V^{(A_1)}$, $V^{(A_2)}$, $V^{(B_1)}$ and $V^{(B_2)}$ satisfying (see equation (31) of [4])

$$\frac{\mathrm{d}V^{(A_1)}}{\mathrm{d}\eta}(\eta = 0) = 0 \qquad \frac{\mathrm{d}V^{(A_1)}}{\mathrm{d}\eta}(\eta = \pi/2) = 0 \qquad (18a)$$

$$V^{(A_2)}(\eta = 0) = 0 \qquad V^{(A_2)}(\eta = \pi/2) = 0 \tag{18b}$$

$$\frac{\mathrm{d}V^{(B_1)}}{\mathrm{d}\eta}(\eta = 0) = 0 \qquad V^{(B_1)}(\eta = \pi/2) = 0 \tag{18c}$$

$$V^{(B_2)}(\eta = 0) = 0 \qquad \frac{\mathrm{d}V^{(B_2)}}{\mathrm{d}\eta}(\eta = \pi/2) = 0. \tag{18d}$$

3.2. Eigenfunctions of the whispering-gallery type

The resonant modes $\Psi(\xi, \eta)$ of the cavity Ω and the associated resonances k are the solutions of the Helmholtz equation (in elliptic coordinates)

$$\frac{\partial^2\Psi}{\partial\xi^2} + \frac{\partial^2\Psi}{\partial\eta^2} + (kc)^2(\cosh^2\xi - \cos^2\eta)\Psi = 0 \tag{19}$$

subject to the Dirichlet boundary condition at $\xi = \xi_0$. Since the coordinate system (16), as well as the Helmholtz equation (19), are invariant under the transformation $(\xi, \eta) \to (-\xi, -\eta)$, the resonant modes $\Psi(\xi, \eta)$ are also subject to the condition

$$\Psi(-\xi, -\eta) = \Psi(\xi, \eta). \tag{20}$$

These mode solutions are sought by separation of variables, in the form $\Psi(\xi, \eta) = U(\xi)V(\eta)$, where $U(\xi)$ and $V(\eta)$ must satisfy the modified Mathieu equation

$$U''(\xi) - (kc)^2(b^2 - \cosh^2\xi)U(\xi) = 0 \tag{21}$$

and the (ordinary) Mathieu equation

$$V''(\eta) + (kc)^2(b^2 - \cos^2\eta)V(\eta) = 0 \tag{22}$$

respectively. The symmetry of the modified Mathieu equation (21) under the exchange $\xi \to -\xi$ leads us to consider two linearly independent solutions: an even solution, denoted by $Uc(\xi, kc, b)$ and an odd one, denoted by $Us(\xi, kc, b)$. Moreover, these two solutions are assumed to be regular at $\xi = 0$. Without loss of generality, we require that they satisfy the boundary conditions

$$\begin{aligned} &Uc(0, kc, b) = 1 \qquad Uc'(0, kc, b) = 0 \\ &Us(0, kc, b) = 0 \qquad Us'(0, kc, b) = 1 \end{aligned} \tag{23}$$

with $Uc' = \mathrm{d}Uc/\mathrm{d}\xi$ and $Us' = \mathrm{d}Us/\mathrm{d}\xi$. Similarly, because of symmetry under the exchange $\eta \to -\eta$, we consider two linearly independent solutions of the Mathieu equation (22), denoted by $c(\eta, kc, b)$ (even solution) and $s(\eta, kc, b)$ (odd solution), normalized in such a way that

$$\begin{aligned} &c(0, kc, b) = 1 \qquad c'(0, kc, b) = 0 \\ &s(0, kc, b) = 0 \qquad s'(0, kc, b) = 1 \end{aligned} \tag{24}$$

with $c' = \mathrm{d}c/\mathrm{d}\eta$ and $s' = \mathrm{d}s/\mathrm{d}\eta$. The combination of these different solutions leads us to seek the resonant modes in the form

$$\Psi^{(c)}(\xi, \eta, kc, b) = Uc(\xi, kc, b)c(\eta, kc, b) \tag{25a}$$

$$\Psi^{(s)}(\xi, \eta, kc, b) = Us(\xi, kc, b)s(\eta, kc, b). \tag{25b}$$

Indeed, the combinations $Uc(\xi, kc, b)s(\eta, kc, b)$ and $Us(\xi, kc, b)c(\eta, kc, b)$ are excluded from the condition (20).

We first assume that $\eta \in]-\infty, +\infty[$. The Dirichlet boundary condition reads

$$Uc(\xi_0, kc, b) = 0 \qquad \forall kc \tag{26a}$$

$$Us(\xi_0, kc, b) = 0 \qquad \forall kc. \tag{26b}$$

We thus determine two sets of eigenvalues, labelled $\{b_\ell^{(c)}(kc)\}_{\ell\in\mathbb{N}^*}$ and $\{b_\ell^{(s)}(kc)\}_{\ell\in\mathbb{N}^*}$, which satisfy respectively equations (26*a*) and (26*b*).

Resonances are the particular values of the reduced wavenumber kc for which the functions $c_\ell(\eta, kc) \stackrel{\text{def}}{=} c(\eta, kc, b_\ell^{(c)})$ and $s_\ell(\eta, kc) \stackrel{\text{def}}{=} s(\eta, kc, b_\ell^{(s)})$ satisfy the symmetry properties (18*a*)–(18*d*). The conditions for resonance appear then as

$$c'_\ell(\pi/2, kc) = 0 \qquad (A_1) \tag{27a}$$

$$s_\ell(\pi/2, kc) = 0 \qquad (A_2) \tag{27b}$$

$$c_\ell(\pi/2, kc) = 0 \qquad (B_1) \tag{27c}$$

$$s'_\ell(\pi/2, kc) = 0 \qquad (B_2). \tag{27d}$$

By solving these equations, we obtain four sets of resonances labelled $(kc)_{n,\ell}^{(A_1)}$, $(kc)_{n,\ell}^{(A_2)}$, $(kc)_{n,\ell}^{(B_1)}$ and $(kc)_{n,\ell}^{(B_2)}$ with n even (respectively n odd) in the representations A_1 and A_2 (respectively the representations B_1 and B_2). Finally, the whispering-gallery modes (corresponding to $\ell = 1$) can be constructed in the form

$$\Psi_{2r}^{(A_1)}(\xi, \eta) = Uc(\xi, (kc)_{2r,1}^{(A_1)}, b_1^{(c)})c(\eta, (kc)_{2r,1}^{(A_1)}, b_1^{(c)}) \qquad r \in \mathbb{N} \tag{28a}$$

$$\Psi_{2r}^{(A_2)}(\xi, \eta) = Us(\xi, (kc)_{2r,1}^{(A_2)}, b_1^{(s)})s(\eta, (kc)_{2r,1}^{(A_2)}, b_1^{(s)}) \qquad r \in \mathbb{N}^* \tag{28b}$$

$$\Psi_{2r+1}^{(B_1)}(\xi, \eta) = Uc(\xi, (kc)_{2r+1,1}^{(B_1)}, b_1^{(c)})c(\eta, (kc)_{2r+1,1}^{(B_1)}, b_1^{(c)}) \qquad r \in \mathbb{N} \tag{28c}$$

$$\Psi_{2r+1}^{(B_2)}(\xi, \eta) = Us(\xi, (kc)_{2r+1,1}^{(B_2)}, b_1^{(s)})s(\eta, (kc)_{2r+1,1}^{(B_2)}, b_1^{(s)}) \qquad r \in \mathbb{N}. \tag{28d}$$

It should be noted that each resonance is non-degenerate. The symmetry breaking $\mathrm{O}(2) \longrightarrow \mathcal{C}_{2\mathrm{v}}$ leads to the splitting up of resonances.

3.3. Asymptotic expansions for the eigenvalues b_ℓ

In order to determine the asymptotic expansions (for $kc \to +\infty$) for the eigenvalues $b_\ell^{(c)}(kc)$ and $b_\ell^{(s)}(kc)$, we need the uniform asymptotic expansions for the radial solutions $Uc(\xi, kc, b)$ and $Us(\xi, kc, b)$. They can be constructed by using the Langer–Olver method [7, 8], which has been previously applied to the Mathieu equations [4].

For large kc, the Langer–Olver method permits us to obtain a pair of linearly independent solutions of the modified Mathieu equation (21) in the form

$$U_+(\xi, b) = \mathrm{e}^{-\mathrm{i}\pi/2}\zeta^{1/4}(\mathrm{b}^2 - \cosh^2\xi)^{-1/4} \times\left\{\mathrm{Ai}[(kc)^{2/3}\zeta]\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{(kc)^{2s}} + \frac{\mathrm{Ai}'[(kc)^{2/3}\zeta]}{(kc)^{4/3}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{(kc)^{2s}}\right\} \tag{29}$$

and

$$U_-(\xi, b) = \mathrm{e}^{-\mathrm{i}\pi/2}\zeta^{1/4}(b^2 - \cosh^2\xi)^{-1/4} \times\left\{\mathrm{Ai}[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta]\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{(kc)^{2s}} + \frac{\mathrm{e}^{2\mathrm{i}\pi/3}\mathrm{Ai}'[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta]}{(kc)^{4/3}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{(kc)^{2s}}\right\} \tag{30}$$

where $\zeta = \zeta(\xi, b)$ is defined by

$$\tfrac{2}{3}(\pm\zeta)^{3/2} = \mp\int_{\mathrm{arccosh}b}^{\xi}[\pm(b^2 - \cosh^2 u)]^{1/2}\,\mathrm{d}u \tag{31}$$

and where the coefficients $A_s(\zeta)$ and $B_s(\zeta)$ are given by recurrence relations. The two functions $Uc(\xi, b)$ and $Us(\xi, b)$, subject to the conditions (23), can be constructed as linear combinations of $U_+(\xi, b)$ and $U_-(\xi, b)$. We obtain the uniform asymptotic expansions, valid for large kc and $\xi \simeq \operatorname{arccosh} b$,

$$Uc(\xi, b) = \frac{2\pi \mathrm{e}^{\mathrm{i}\pi/6}}{(kc)^{2/3}} \{U'_-(0, b)U_+(\xi, b) - U'_+(0, b)U_-(\xi, b)\} \tag{32}$$

and

$$Us(\xi, b) = -\frac{2\pi \mathrm{e}^{\mathrm{i}\pi/6}}{(kc)^{2/3}} \{U_-(0, b)U_+(\xi, b) - U_+(0, b)U_-(\xi, b)\}. \tag{33}$$

From (32) and (33), equations (26*a*) and (26*b*) become respectively

$$U'_-(0, b)U_+(\xi_0, b) = U'_+(0, b)U_-(\xi_0, b) \tag{34a}$$

which provides the eigenvalues $b_\ell^{(c)}$, and

$$U_-(0, b)U_+(\xi_0, b) = U_+(0, b)U_-(\xi_0, b) \tag{34b}$$

which provides the eigenvalues $b_\ell^{(s)}$. It can be shown (see appendix B) that the right-hand sides in equations (34*a*) and (34*b*) are exponentially small with respect to the left-hand ones. Thus, in a first approximation, we consider that (34*a*) and (34*b*) both reduce to the single equation

$$U_+(\xi_0, b) = 0. \tag{35}$$

Equation (35) does not permit us to determine the two sets of eigenvalues, but only one, which is denoted by $\{\overline{b}_\ell\}_{\ell \in \mathbb{N}^*}$. We only consider the first two leading terms (corresponding to $s = 0$) of the asymptotic expansion (29) at $\xi = \xi_0$:

$$\begin{aligned} U_+(\xi_0, b) = \mathrm{e}^{-\mathrm{i}\pi/2}[\zeta(\xi_0, b)]^{1/4}(b^2 - \cosh^2\xi_0)^{-1/4} \Bigg\{ & \mathrm{Ai}[(kc)^{2/3}\zeta(\xi_0, b)] \\ & + B_0[\zeta(\xi_0, b)] \frac{\mathrm{Ai}'[(kc)^{2/3}\zeta(\xi_0, b)]}{(kc)^{4/3}} \Bigg\} \left[1 + \underset{kc\to+\infty}{\mathrm{O}} \left(\frac{1}{(kc)^2}\right)\right] \end{aligned} \tag{36}$$

with

$$\begin{aligned} B_0(\zeta) = -\frac{5}{48\zeta^2} + \frac{\sqrt{b^2 - 1}}{12\sqrt{\zeta}} \Bigg\{ & \frac{\mathrm{i}F(\mathrm{i}\xi|\frac{1}{1-b^2})}{(b^2 - 1)} - \frac{\mathrm{i}(2b^4 - 3b^2 + 1)E(\mathrm{i}\xi|\frac{1}{1-b^2})}{2\,b^2(b^2 - 1)^2} \\ & + \frac{(14b^4 - 14b^2 + 1)\sinh 2\xi + (b^2 - 1/2)\sinh 4\xi}{8\,b^2(b^2 - 1)^2(b^2 - \cosh^2\xi)^{3/2}} \Bigg\}. \end{aligned} \tag{37}$$

Here, F and E denote the elliptic integrals of the first and second kind respectively [21], defined by

$$F(\phi|m) = \int_0^\phi (1 - m\sin^2\theta)^{-1/2}\,\mathrm{d}\theta \qquad \text{and} \qquad E(\phi|m) = \int_0^\phi (1 - m\sin^2\theta)^{1/2}\,\mathrm{d}\theta. \tag{38}$$

To this degree of approximation, the equation $U_+(\xi_0, b) = 0$ can be written as

$$\frac{\mathrm{Ai}[(kc)^{2/3}\zeta(\xi_0, b)]}{\mathrm{Ai}'[(kc)^{2/3}\zeta(\xi_0, b)]} \approx -(kc)^{-4/3}B_0[\zeta(\xi_0, b)]. \tag{39}$$

Equation (39) is solved perturbatively by following the method previously developed in the context of the circular cavity (see equations (10)–(13)) and we obtain, for the eigenvalues, the asymptotic expansions:

$$\overline{b}_\ell(kc) = \cosh\xi_0 + \frac{2^{-1/3}(\sinh\xi_0)^{2/3}}{(\cosh\xi_0)^{1/3}} q_{1,0}(x_\ell)(kc)^{-2/3}$$

$$
\begin{aligned}
&+\frac{2^{-2/3}}{60(\cosh\xi_0)^{5/3}(\sinh\xi_0)^{2/3}}\left[\sum_{j=0}^{1} q_{2,j}(x_\ell)\cosh(2j\xi_0)\right](kc)^{-4/3}\\
&+\frac{1}{16\,800(\cosh\xi_0)^{3}(\sinh\xi_0)^{2}}\left[\sum_{j=0}^{2} q_{3,j}(x_\ell)\cosh(2j\xi_0)\right](kc)^{-2}\\
&+\frac{2^{-1/3}}{36\,288\,000(\cosh\xi_0)^{13/3}(\sinh\xi_0)^{10/3}}\left[\sum_{j=0}^{3} q_{4,j}(x_\ell)\cosh(2j\xi_0)\right](kc)^{-8/3}\\
&+\frac{2^{-2/3}}{167\,650\,560\,000(\cosh\xi_0)^{17/3}(\sinh\xi_0)^{14/3}}\\
&\times\left[\sum_{j=0}^{4} q_{5,j}(x_\ell)\cosh(2j\xi_0)\right](kc)^{-10/3}\\
&+\frac{1}{6457\,651\,200\,000(\cosh\xi_0)^{7}(\sinh\xi_0)^{6}}\left[\sum_{j=0}^{5} q_{6,j}(x_\ell)\cosh(2j\xi_0)\right](kc)^{-4}\\
&+\underset{kc\to+\infty}{\mathrm{O}}[(kc)^{-14/3}].
\end{aligned}
\tag{40}
$$

Here, the $q_{i,j}(x_\ell)$ are polynomials of degree i in x_ℓ which have been already obtained in the scattering problem [4], and are given by

$$q_{1,0}(x_\ell) = x_\ell \tag{41a}$$

$$
\begin{aligned}
q_{2,0}(x_\ell) &= 15\,x_\ell^{\,2}\\
q_{2,1}(x_\ell) &= x_\ell^{\,2}
\end{aligned}
\tag{41b}
$$

$$
\begin{aligned}
q_{3,0}(x_\ell) &= 570 + 407\,x_\ell^{\,3}\\
q_{3,1}(x_\ell) &= -980\,x_\ell^{\,3}\\
q_{3,2}(x_\ell) &= -3(10 + x_\ell^{\,3})
\end{aligned}
\tag{41c}
$$

$$
\begin{aligned}
q_{4,0}(x_\ell) &= 90\,(6840\,x_\ell + 13\,711\,x_\ell^{\,4})\\
q_{4,1}(x_\ell) &= -21\,(119\,880\,x_\ell + 48\,037\,x_\ell^{\,4})\\
q_{4,2}(x_\ell) &= -90\,(360\,x_\ell - 5641\,x_\ell^{\,4})\\
q_{4,3}(x_\ell) &= 10\,440\,x_\ell + 281\,x_\ell^{\,4}
\end{aligned}
\tag{41d}
$$

$$
\begin{aligned}
q_{5,0}(x_\ell) &= 3\,(4557\,365\,100\,x_\ell^{\,2} + 1125\,295\,351\,x_\ell^{\,5})\\
q_{5,1}(x_\ell) &= -27\,720\,(283\,050\,x_\ell^{\,2} + 266\,977\,x_\ell^{\,5})\\
q_{5,2}(x_\ell) &= 4\,(1919\,325\,600\,x_\ell^{\,2} + 481\,897\,921\,x_\ell^{\,5})\\
q_{5,3}(x_\ell) &= 9240\,(8550\,x_\ell^{\,2} - 60\,997\,x_\ell^{\,5})\\
q_{5,4}(x_\ell) &= -(6624\,900\,x_\ell^{\,2} + 73\,769\,x_\ell^{\,5})
\end{aligned}
\tag{41e}
$$

$$
\begin{aligned}
q_{6,0}(x_\ell) &= 5850\,(318\,120 + 33\,314\,840\,x_\ell^{\,3} + 15\,814\,949\,x_\ell^{\,6})\\
q_{6,1}(x_\ell) &= -10\,(1491\,919\,000 + 63\,787\,753\,000\,x_\ell^{\,3} + 10\,704\,688\,123\,x_\ell^{\,6})\\
q_{6,2}(x_\ell) &= -2600\,(75\,240 - 48\,704\,080\,x_\ell^{\,3} - 27\,746\,557\,x_\ell^{\,6})\\
q_{6,3}(x_\ell) &= 35\,(13\,413\,400 - 2239\,211\,400\,x_\ell^{\,3} - 382\,250\,049\,x_\ell^{\,6})\\
q_{6,4}(x_\ell) &= 650\,(3960 - 886\,040\,x_\ell^{\,3} + 4013\,887\,x_\ell^{\,6})\\
q_{6,5}(x_\ell) &= -(1744\,600 - 16\,495\,400\,x_\ell^{\,3} - 93\,617\,x_\ell^{\,6}).
\end{aligned}
\tag{41f}
$$

It should be noted that the expression (40) can be recovered from the asymptotic expansions for the Regge poles in the scattering problem (see [4], equation (56)) by changing x_ℓ in $e^{2i\pi/3}x_\ell$.

3.4. Exponentially improved asymptotic expansions for eigenvalues b_ℓ

We have obtained the perturbative part of the asymptotic expansions for the eigenvalues $b_\ell^{(c)}$ and $b_\ell^{(c)}$. We now need to capture the exponentially small terms describing the splitting up of these eigenvalues. Thus, we again consider equations (34*a*) and (34*b*) but now we take into account the exponentially small contributions. From the uniform asymptotic expansions

$$\begin{aligned} Uc(\xi, b) = {} & 2\pi \mathrm{e}^{\mathrm{i}\pi/6} \zeta(0,b)^{-1/4} \zeta(\xi,b)^{1/4} (b^2-1)^{1/4} (b^2 - \cosh^2\xi)^{-1/4} \\ & \times \Big\{ \mathrm{e}^{2\mathrm{i}\pi/3} \mathrm{Ai}'[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0,b)] \mathrm{Ai}[(kc)^{2/3}\zeta(\xi,b)] \\ & - \mathrm{Ai}'[(kc)^{2/3}\zeta(0,b)] \mathrm{Ai}[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\xi,b)] \Big\} [1 + \underset{kc\to+\infty}{\mathrm{O}}(1/kc)] \end{aligned} \tag{42}$$

$$\begin{aligned} Us(\xi, b) = {} & 2\pi \mathrm{e}^{\mathrm{i}\pi/6} (kc)^{-2/3} \zeta(0,b)^{1/4} \zeta(\xi,b)^{1/4} (b^2-1)^{-1/4} (b^2 - \cosh^2\xi)^{-1/4} \\ & \times \{ \mathrm{Ai}[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0,b)] \mathrm{Ai}[(kc)^{2/3}\zeta(\xi,b)] \\ & - \mathrm{Ai}[(kc)^{2/3}\zeta(0,b)] \mathrm{Ai}[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\xi,b)] \} [1 + \underset{kc\to+\infty}{\mathrm{O}}(1/kc)] \end{aligned} \tag{43}$$

and by using the asymptotic behaviour of the Airy function (see appendix A), equations (34*a*) (providing the $b_\ell^{(c)}$ associated with the 'even' modes) and (34*b*) (providing the $b_\ell^{(s)}$ associated with the 'odd' modes) become respectively

$$\mathrm{Ai}[(kc)^{2/3}\zeta(\xi_0,b)] \approx -\mathrm{e}^{\mathrm{i}\pi/6} \exp[-\tfrac{4}{3}kc[\zeta(0,b)]^{3/2}] \mathrm{Ai}[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\xi_0,b)] \tag{44a}$$

and

$$\mathrm{Ai}[(kc)^{2/3}\zeta(\xi_0,b)] \approx \mathrm{e}^{\mathrm{i}\pi/6} \exp[-\tfrac{4}{3}kc[\zeta(0,b)]^{3/2}] \mathrm{Ai}[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\xi_0,b)]. \tag{44b}$$

Neglecting the exponentially small contributions in the previous two equations, reduce (44*a*) and (44*b*) to the single equation (11). Therefore, equation (39) can be exponentially improved and leads to

$$\begin{aligned} \frac{\mathrm{Ai}[(kc)^{2/3}\zeta(\xi_0,b)]}{\mathrm{Ai}'[(kc)^{2/3}\zeta(\xi_0,b)]} = - \Bigg\{ & (kc)^{-4/3} B_0[\zeta(\xi_0,b)] + \underbrace{\cdots\cdots\cdots\cdots}_{\text{higher orders in } 1/(kc)} \\ & \underbrace{\pm\, \mathrm{e}^{\mathrm{i}\pi/6} \frac{\mathrm{Ai}[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\xi_0,b)]}{\mathrm{Ai}'[(kc)^{2/3}\zeta(\xi_0,b)]} \exp\left[-\frac{4}{3}kc[\zeta(0,b)]^{3/2} \right]}_{\text{term lying beyond all orders}} \Bigg\} \end{aligned} \tag{45}$$

where the upper and lower signs correspond respectively to the even (c) and odd (s) cases. By expanding equation (45) in a Taylor series about the solutions $b = \overline{b}_\ell$ of equation (39), and then by using the approximation $(kc)^{2/3}\zeta(\xi_0, \overline{b}_\ell) \approx x_\ell$, we obtain the eigenvalues $b_\ell^{(c)}$ and $b_\ell^{(s)}$ in the form

$$b_\ell^{(c)}(kc) = \overline{b}_\ell(kc) + \tfrac{1}{2}\delta b_\ell(kc) \tag{46a}$$

$$b_\ell^{(s)}(kc) = \overline{b}_\ell(kc) - \tfrac{1}{2}\delta b_\ell(kc) \tag{46b}$$

where $\overline{b}_\ell(kc)$ is given by (40), and where

$$\delta b_\ell(kc) = b_\ell^{(c)}(kc) - b_\ell^{(s)}(kc) = -\frac{2\mathrm{e}^{\mathrm{i}\pi/6}\mathrm{Ai}(\mathrm{e}^{2\mathrm{i}\pi/3}x_\ell)}{(kc)^{2/3}(\partial\zeta/\partial b)_{\xi_0,\overline{b}_\ell}\mathrm{Ai}'(x_\ell)} \exp\left[-\frac{4}{3}kc[\zeta(0,\overline{b}_\ell)]^{3/2} \right]. \tag{47}$$

The splitting described by (47) is purely of exponential nature since higher orders in $1/(kc)$ have been neglected in equations (42) and (43). It would be possible to take into account these corrections, but it is clear that their contributions are insignificant.

Table 2. Average values and splitting of eigenvalues b_1 for $\xi_0 = 1$ and various kc.

	$\overline{b}_1$		δb_1		$\ln(\delta b_1)$	
kc	Exact	Asympt.	Exact	Asympt.	Exact	Asympt.
6	1.069 453	1.068 447	2.466×10^{-2}	2.711×10^{-2}	−3.702	−3.608
7	1.106 977	1.105 920	8.493×10^{-3}	9.112×10^{-3}	−4.769	−4.698
8	1.137 940	1.137 362	2.586×10^{-3}	2.708×10^{-3}	−5.958	−5.912
9	1.164 278	1.163 976	7.184×10^{-4}	7.419×10^{-4}	−7.238	−7.206
10	1.186 927	1.186 759	1.868×10^{-4}	1.915×10^{-4}	−8.585	−8.561
11	1.206 579	1.206 479	4.625×10^{-5}	4.719×10^{-5}	−9.981	−9.961
12	1.223 787	1.223 724	1.102×10^{-5}	1.121×10^{-5}	−11.416	−11.399
13	1.238 986	1.238 944	2.543×10^{-6}	2.580×10^{-6}	−12.882	−12.868
14	1.252 517	1.252 489	5.713×10^{-7}	5.786×10^{-7}	−14.375	−14.363
15	1.264 649	1.264 629	1.255×10^{-7}	1.269×10^{-7}	−15.891	−15.880
16	1.275 597	1.275 583	2.702×10^{-8}	2.730×10^{-8}	−17.427	−17.417
17	1.285 533	1.285 523	5.717×10^{-9}	5.771×10^{-9}	−18.980	−18.970

Table 3. Average values and splitting of eigenvalues b_1 for $kc = 10$ and various ξ_0.

	$\overline{b}_1$		δb_1		$\ln(\delta b_1)$	
ξ_0	Exact	Asympt.	Exact	Asympt.	Exact	Asympt.
0.8	1.048 023	1.047 363	1.261×10^{-2}	1.369×10^{-2}	−4.373	−4.291
0.9	1.108 630	1.108 171	2.229×10^{-3}	2.327×10^{-3}	−6.106	−6.063
1.0	1.186 927	1.186 759	1.868×10^{-4}	1.915×10^{-4}	−8.585	−8.561
1.1	1.283 353	1.283 283	7.292×10^{-6}	7.425×10^{-6}	−11.829	−11.811
1.2	1.398 411	1.398 379	1.243×10^{-7}	1.262×10^{-7}	−15.901	−15.885
1.3	1.533 028	1.533 012	8.412×10^{-10}	8.518×10^{-10}	−20.896	−20.884
1.4	1.688 434	1.688 426	2.000×10^{-12}	2.022×10^{-12}	−26.938	−26.927

'Exact' values for $b_\ell^{(c)}$ and $b_\ell^{(s)}$ can be obtained by numerically solving equations (26*a*) and (26*b*). Exact average eigenvalues and splitting are then defined as $\overline{b}_\ell = \frac{1}{2}(b_\ell^{(c)} + b_\ell^{(s)})$ and $\delta b_\ell = b_\ell^{(c)} - b_\ell^{(s)}$ respectively. In table 2 and 3, the asymptotic approximations (40) and (47) are compared with the exact values for $\overline{b}_1$ and δb_1 defined above. A good agreement is found, and the asymptotic expansions give even better approximations when kc and ξ_0 are large. Furthermore, it should be noted that the splitting is always well captured and is numerically significant for low frequencies.

3.5. Asymptotic expansions for resonances

Resonances are the particular values of the reduced wavenumber kc which satisfy equations (27*a*)–(27*d*). First, we determine the asymptotic expansions for resonances without paying attention to the terms lying beyond all orders. In order to do so, we solve (27*a*)–(27*d*) by using the WKB expansions [4]

$$c(\eta, b) = \frac{1}{2} R(0,b)^{1/2} R(\eta,b)^{-1/2} \times\left\{ \exp\left[kc\int_0^\eta R(\eta',b)\,\mathrm{d}\eta'\right]\left(1+\frac{Y_1(\eta,b)}{kc}+\underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right) + \exp\left[-kc\int_0^\eta R(\eta',b)\,\mathrm{d}\eta'\right]\left(1-\frac{Y_1(\eta,b)}{kc}+\underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right)\right\} \tag{48}$$

$$s(\eta, b) = \frac{1}{2kc} R(0,b)^{-1/2} R(\eta,b)^{-1/2}$$
$$\times \left\{ \exp\left[kc \int_0^\eta R(\eta', b)\,\mathrm{d}\eta'\right] \left(1 + \frac{Y_1(\eta,b)}{kc} + \underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right)\right.$$
$$\left. - \exp\left[-kc \int_0^\eta R(\eta', b)\,\mathrm{d}\eta'\right] \left(1 - \frac{Y_1(\eta,b)}{kc} + \underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right)\right\} \tag{49}$$

$$c'(\eta, b) = \frac{kc}{2} R(0,b)^{1/2} R(\eta,b)^{1/2}$$
$$\times \left\{ \exp\left[kc \int_0^\eta R(\eta', b)\,\mathrm{d}\eta'\right] \left(1 + \frac{1}{kc}\left(Y_1(\eta,b) - \frac{R'(\eta,b)}{2R^2(\eta,b)}\right)\right.\right.$$
$$\left. + \underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right) - \exp\left[-kc \int_0^\eta R(\eta', b)\,\mathrm{d}\eta'\right]$$
$$\left. \times \left(1 - \frac{1}{kc}\left(Y_1(\eta,b) - \frac{R'(\eta,b)}{2R^2(\eta,b)}\right) + \underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right)\right\} \tag{50}$$

and

$$s'(\eta, b) = \tfrac{1}{2} R(0,b)^{-1/2} R(\eta,b)^{1/2}$$
$$\times \left\{ \exp\left[kc \int_0^\eta R(\eta', b)\,\mathrm{d}\eta'\right] \left(1 + \frac{1}{kc}\left(Y_1(\eta,b) - \frac{R'(\eta,b)}{2R^2(\eta,b)}\right)\right.\right.$$
$$\left. + \underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right) + \exp\left[-kc \int_0^\eta R(\eta', b)\,\mathrm{d}\eta'\right]$$
$$\left. \times \left(1 - \frac{1}{kc}\left(Y_1(\eta,b) - \frac{R'(\eta,b)}{2R^2(\eta,b)}\right) + \underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right)\right)\right\} \tag{51}$$

with

$$R(\eta, b) = \mathrm{i}(b^2 - \cos^2\eta)^{1/2} \tag{52}$$

and

$$Y_1(\eta, b) = \int_0^\eta \left(\frac{R''(\eta', b)}{4R^2(\eta', b)} - \frac{3R'^2(\eta', b)}{8R^3(\eta', b)}\right)\mathrm{d}\eta'. \tag{53}$$

The four conditions for resonance then reduce to

$$\Phi(kc, \overline{b}_\ell) = \frac{n\pi}{2} - \frac{Y_1(\pi/2, \overline{b}_\ell)}{\mathrm{i}\,kc} + \underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{(kc)^2}\right) \tag{54}$$

with $n \in \mathbb{N}$. Resonances A_1 and A_2 (respectively resonances B_1 and B_2) correspond to n even (respectively to n odd). Here, the functions $\Phi(kc, b)$ and $Y_1(\pi/2, b)$ are given by

$$\Phi(kc, b) = kc \int_0^{\pi/2} (b^2 - \cos^2\eta)^{1/2}\,\mathrm{d}\eta = kc\sqrt{b^2 - 1}\,E[1/(1-b^2)] \tag{55}$$

and

$$Y_1(\pi/2, b) = \frac{(1 - 2b^2)E[1/(1-b^2)] + 2b^2 K[1/(1-b^2)]}{24b^2\sqrt{1-b^2}} \tag{56}$$

where K and E denote complete elliptic integrals of the first and second kinds respectively [21], defined by (see also (38))

$$K(m) = F(\pi/2|m) = \int_0^{\pi/2} (1 - m\sin^2\theta)^{-1/2}\,\mathrm{d}\theta \tag{57a}$$

and

$$E(m) = E(\pi/2|m) = \int_0^{\pi/2} (1 - m\sin^2\theta)^{1/2}\,\mathrm{d}\theta. \tag{57b}$$

By taking for $\overline{b}_\ell(kc)$ its asymptotic expansion (40), we can solve (54) perturbatively, and we obtain

$$\begin{aligned}(\overline{kc})_{n,\ell} = {} & \frac{\pi}{2\tilde{\mathrm{E}}(\xi_0)}\, n - \frac{\pi^{1/3} x_\ell\, \tilde{\mathrm{K}}(\xi_0)}{2^{2/3}\,\tilde{\mathrm{E}}^{4/3}(\xi_0)(\cosh\xi_0\,\sinh\xi_0)^{1/3}}\, n^{1/3} \\ & + \frac{x_\ell^2\, Q_2(\xi_0, x_\ell)}{2^{1/3}\,\pi^{1/3}\, 60\,\tilde{\mathrm{E}}^{5/3}(\xi_0)(\cosh\xi_0\,\sinh\xi_0)^{5/3}}\, n^{-1/3} \\ & + \frac{Q_3(\xi_0, x_\ell)}{8400\,\pi\,\tilde{\mathrm{E}}^{2}(\xi_0)(\cosh\xi_0\,\sinh\xi_0)^{3}}\, n^{-1} \\ & - \frac{x_\ell\, Q_4(\xi_0, x_\ell)}{2^{2/3}\,\pi^{5/3}\, 4536\,000\,\tilde{\mathrm{E}}^{7/3}(\xi_0)(\cosh\xi_0\,\sinh\xi_0)^{13/3}}\, n^{-5/3} \\ & + \frac{x_\ell^2\, Q_5(\xi_0, x_\ell)}{2^{1/3}\,\pi^{7/3}\, 41\,912\,640\,000\,\tilde{\mathrm{E}}^{8/3}(\xi_0)(\cosh\xi_0\,\sinh\xi_0)^{17/3}}\, n^{-7/3} \\ & + \underset{n\to+\infty}{\mathrm{O}}(n^{-3}) \end{aligned} \tag{58}$$

where the polynomials $Q_i(x_\ell)$ are given in [4] (equation (63)). In the previous formula, we have defined $\tilde{\mathrm{K}}(\xi_0) = \cosh\xi_0\, K(-1/\sinh^2\xi_0)$ and $\tilde{\mathrm{E}}(\xi_0) = \sinh\xi_0\, E(-1/\sinh^2\xi_0)$. It should be noted that the previous expression can be recovered from equation (62) of [4] by changing again x_ℓ in $\mathrm{e}^{2\mathrm{i}\pi/3}x_\ell$.

3.6. Exponentially improved asymptotic expansions for resonances

In order to display the splitting up of resonances, we use the uniform asymptotic expansions for the functions $c(\eta, b)$, $s(\eta, b)$, $c'(\eta, b)$ and $s'(\eta, b)$. They have been constructed, by applying the Langer–Olver method, in [4]. The leading terms of these uniform asymptotic expansions, valid for large kc and $\eta \simeq \arccos b$, are given by

$$\begin{aligned} c(\eta, b) = {} & -2\pi\mathrm{e}^{\mathrm{i}\pi/6}\vartheta(0,b)^{-1/4}\vartheta(\eta,b)^{1/4}(b^2-1)^{1/4}(b^2-\cos^2\eta)^{-1/4} \\ & \times[\mathrm{Ai}'((kc)^{2/3}\vartheta(0,b))\mathrm{Ai}(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\vartheta(\eta,b)) \\ & -\mathrm{e}^{2\mathrm{i}\pi/3}\mathrm{Ai}'(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\vartheta(0,b))\mathrm{Ai}((kc)^{2/3}\vartheta(\eta,b))]\left(1+\underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{kc}\right)\right) \end{aligned} \tag{59}$$

$$\begin{aligned} s(\eta, b) = {} & -2\pi\mathrm{e}^{-\mathrm{i}\pi/3}(kc)^{-2/3}\vartheta(0,b)^{1/4}\vartheta(\eta,b)^{1/4}(b^2-1)^{-1/4}(b^2-\cos^2\eta)^{-1/4} \\ & \times[\mathrm{Ai}((kc)^{2/3}\vartheta(0,b))\mathrm{Ai}(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\vartheta(\eta,b)) \\ & -\mathrm{Ai}(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\vartheta(0,b))\mathrm{Ai}((kc)^{2/3}\vartheta(\eta,b))]\left(1+\underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{kc}\right)\right) \end{aligned} \tag{60}$$

$$\begin{aligned} c'(\eta, b) = {} & 2\pi\mathrm{e}^{4\mathrm{i}\pi/3}(kc)^{2/3}\vartheta(0,b)^{-1/4}\vartheta(\eta,b)^{-1/4}(b^2-1)^{1/4}(b^2-\cos^2\eta)^{1/4} \\ & \times[\mathrm{Ai}'((kc)^{2/3}\vartheta(0,b))\mathrm{Ai}'(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\vartheta(\eta,b)) \\ & -\mathrm{Ai}'(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\vartheta(0,b))\mathrm{Ai}'((kc)^{2/3}\vartheta(\eta,b))]\left(1+\underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{kc}\right)\right) \end{aligned} \tag{61}$$

and

$$\begin{aligned} s'(\eta, b) = {} & 2\pi\mathrm{e}^{\mathrm{i}\pi/6}\vartheta(0,b)^{1/4}\vartheta(\eta,b)^{-1/4}(b^2-1)^{-1/4}(b^2-\cos^2\eta)^{1/4} \\ & \times[\mathrm{Ai}((kc)^{2/3}\vartheta(0,b))\mathrm{e}^{2\mathrm{i}\pi/3}\mathrm{Ai}'(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\vartheta(\eta,b)) \end{aligned}$$

$$-\mathrm{Ai}(\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\vartheta(0,b))\mathrm{Ai}'((kc)^{2/3}\vartheta(\eta,b))]\left(1+\underset{kc\to+\infty}{\mathrm{O}}\left(\frac{1}{kc}\right)\right) \tag{62}$$

with ϑ defined by the relation

$$\tfrac{2}{3}[\vartheta(\eta,b)]^{3/2} = -\mathrm{i}\int_{\arccos b}^{\eta}(b^2-\cos^2 v)^{1/2}\,\mathrm{d}v. \tag{63}$$

By neglecting the higher orders in $1/(kc)$, we shall capture only the leading contribution to the splitting. The conditions for resonance $c'_\ell(\pi/2) = 0$, $s_\ell(\pi/2) = 0$, $c_\ell(\pi/2) = 0$ and $s'_\ell(\pi/2) = 0$, written in terms of the previous uniform asymptotic expansions, reduce respectively to

$$\mathrm{Ai}'[(kc)^{2/3}\,\vartheta(0,b_\ell^{(c)})]\mathrm{Ai}'[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\,\vartheta(\pi/2,b_\ell^{(c)})] \approx \mathrm{Ai}'[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\,\vartheta(0,b_\ell^{(c)})]\mathrm{Ai}'[(kc)^{2/3}\,\vartheta(\pi/2,b_\ell^{(c)})] \tag{64a}$$

$$\mathrm{Ai}[(kc)^{2/3}\,\vartheta(0,b_\ell^{(s)})]\mathrm{Ai}[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\,\vartheta(\pi/2,b_\ell^{(s)})] \approx \mathrm{Ai}[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\,\vartheta(0,b_\ell^{(s)})]\mathrm{Ai}[(kc)^{2/3}\,\vartheta(\pi/2,b_\ell^{(s)})] \tag{64b}$$

$$\mathrm{Ai}'[(kc)^{2/3}\,\vartheta(0,b_\ell^{(c)})]\mathrm{Ai}[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\,\vartheta(\pi/2,b_\ell^{(c)})] \approx \mathrm{e}^{2\mathrm{i}\pi/3}\mathrm{Ai}'[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\,\vartheta(0,b_\ell^{(c)})]\mathrm{Ai}[(kc)^{2/3}\,\vartheta(\pi/2,b_\ell^{(c)})] \tag{64c}$$

and

$$\mathrm{Ai}[(kc)^{2/3}\vartheta(0,b_\ell^{(s)})]\mathrm{e}^{2\mathrm{i}\pi/3}\mathrm{Ai}'[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\vartheta(\pi/2,b_\ell^{(s)})] \approx \mathrm{Ai}[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\vartheta(0,b_\ell^{(s)})]\mathrm{Ai}'[(kc)^{2/3}\vartheta(\pi/2,b_\ell^{(s)})]. \tag{64d}$$

In order to express these equations in a simpler way and to take into account Stokes's phenomenon, we shall use the asymptotic behaviour of the Airy functions (see appendix A). For $z = (kc)^{2/3}\,\vartheta(\pi/2,b_\ell)$, $z = \mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\,\vartheta(0,b_\ell)$ and $z = \mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\,\vartheta(\pi/2,b_\ell)$, we take $\sigma = 0$ because $-2\pi/3 < \arg z < 2\pi/3$, while for $z = (kc)^{2/3}\,\vartheta(0,b_\ell)$, we take $\sigma = -1/2$ because z lies exactly on the Stokes line $\arg z = -2\pi/3$. By substituting the suitable asymptotic expansions in equations (64*a*)–(64*d*), and by noting that the function $\Phi(kc,b)$, already defined by (55), can also be expressed from (63) as $\Phi(kc,b) = -\mathrm{i}\frac{2}{3}kc[\vartheta(0,b)^{3/2} - \vartheta(\pi/2,b)^{3/2}]$, we obtain

$$\Phi\left(kc,\overline{b}_\ell \pm \frac{1}{2}\delta b_\ell\right) \approx \frac{n\pi}{2} \pm \frac{1}{4}\exp\left[\frac{4}{3}kc\left[\vartheta\left(0,\overline{b}_\ell \pm \frac{1}{2}\delta b_\ell\right)\right]^{3/2}\right]. \tag{65}$$

By expanding (65) in a Taylor series to the first order, it yields

$$\Phi(kc,\overline{b}_\ell) \approx \frac{n\pi}{2} \pm \left\{\frac{1}{4}\exp\left[\frac{4}{3}kc[\vartheta(0,\overline{b}_\ell)]^{3/2}\right] - \frac{1}{2}\frac{\partial\Phi}{\partial b}(kc,\overline{b}_\ell)\delta b_\ell\right\}. \tag{66}$$

In equations (65) and (66), if n is even, the upper (respectively the lower) sign corresponds to the A_1 (respectively the A_2) representation, while for n odd, the upper (respectively the lower) sign corresponds to the B_1 (respectively the B_2) representation. Finally, from equation (66), the condition for resonance (54) is exponentially improved and reads

$$\Phi[kc,\overline{b}_\ell(kc)] = \frac{n\pi}{2} - \frac{Y_1[\pi/2,\overline{b}_\ell(kc)]}{\mathrm{i}\,kc} + \underbrace{\cdots\cdots\cdots\cdots}_{\text{higher orders in }1/(kc)} \pm \underbrace{\left\{\frac{1}{4}\exp\left[\frac{4}{3}kc\big[\vartheta(0,\overline{b}_\ell(kc))\big]^{3/2}\right] - \frac{1}{2}\frac{\partial\Phi}{\partial b}[kc,\overline{b}_\ell(kc)]\delta b_\ell(kc)\right\}}_{\text{term lying beyond all orders}} \tag{67}$$

with $n \in \mathbb{N}^*$.

The resonances $(kc)_{n,\ell}$ can be sought near the solutions $(\overline{kc})_{n,\ell}$ of equation (54). We can write

$$\begin{aligned} (kc)^{(A_1)}_{2r,\ell} &= (\overline{kc})_{2r,\ell} + \tfrac{1}{2}(\delta kc)_{2r,\ell} \qquad & (kc)^{(B_1)}_{2r+1,\ell} &= (\overline{kc})_{2r+1,\ell} + \tfrac{1}{2}(\delta kc)_{2r+1,\ell} \\ (kc)^{(A_2)}_{2r,\ell} &= (\overline{kc})_{2r,\ell} - \tfrac{1}{2}(\delta kc)_{2r,\ell} \qquad & (kc)^{(B_2)}_{2r+1,\ell} &= (\overline{kc})_{2r+1,\ell} - \tfrac{1}{2}(\delta kc)_{2r+1,\ell} \end{aligned} \tag{68}$$

with $r \in \mathbb{N}^*$ for the A_1 and A_2 resonances, $r \in \mathbb{N}$ for the B_1 and B_2 resonances. The separation $(\delta kc)_{n,\ell}$ appears as the sum of two contributions in the form

$$(\delta kc)_{n,\ell} = (\delta kc)^{\text{rad}}_{n,\ell} + (\delta kc)^{\text{ang}}_{n,\ell}. \tag{69}$$

Here,

$$\begin{aligned} (\delta kc)^{\text{rad}}_{n,\ell} &= -\frac{\frac{\partial\Phi}{\partial b}((\overline{kc})_{n,\ell}, \bar{\bar{b}}_{n,\ell})}{\Phi'((\overline{kc})_{n,\ell})}\delta b_\ell((\overline{kc})_{n,\ell}) \\ &= \frac{2\mathrm{e}^{\mathrm{i}\pi/6}\frac{\partial\Phi}{\partial b}((\overline{kc})_{n,\ell}, \bar{\bar{b}}_{n,\ell})\mathrm{Ai}(\mathrm{e}^{2\mathrm{i}\pi/3}x_\ell)}{(\overline{kc})^{2/3}_{n,\ell}\,\Phi'((\overline{kc})_{n,\ell})\frac{\partial\zeta}{\partial b}(\xi_0, \bar{\bar{b}}_{n,\ell})\mathrm{Ai}'(x_\ell)} \exp\left[-\frac{4}{3}(\overline{kc})_{n,\ell}[\zeta(0, \bar{\bar{b}}_{n,\ell})]^{3/2}\right] \end{aligned} \tag{70}$$

is linked to the splitting up of the eigenvalues b_ℓ of the 'radial' problem, while

$$(\delta kc)^{\text{ang}}_{n,\ell} = \frac{1}{2\Phi'((\overline{kc})_{n,\ell})}\exp\left[\frac{4}{3}(\overline{kc})_{n,\ell}[\vartheta(0, \bar{\bar{b}}_{n,\ell})]^{3/2}\right] \tag{71}$$

arises by taking into account the Stokes phenomenon in the 'angular' problem. In the two previous equations, we have defined

$$\Phi'(kc) = \frac{\mathrm{d}}{\mathrm{d}kc}\Phi(kc, \bar{b}_\ell(kc)) = \frac{\partial\Phi}{\partial kc}(kc, \bar{b}_\ell(kc)) + \frac{\partial\Phi}{\partial b}(kc, \bar{b}_\ell(kc))\frac{\mathrm{d}\bar{b}_\ell}{\mathrm{d}kc}(kc) \tag{72}$$

and

$$\bar{\bar{b}}_{n,\ell} = \bar{b}_\ell((\overline{kc})_{n,\ell}). \tag{73}$$

'Exact' values for the resonances $(kc)_{n,\ell}$ can be obtained by numerically solving equations (27*a*)–(27*d*). Exact average resonances $(\overline{kc})_{n,\ell}$ and splitting $(\delta kc)_{n,\ell}$ are then defined as

$$(\overline{kc})_{n,\ell} = \begin{cases} \frac{1}{2}((kc)^{(A_1)}_{n,\ell} + (kc)^{(A_2)}_{n,\ell}) & (n \text{ even}) \\ \frac{1}{2}((kc)^{(B_1)}_{n,\ell} + (kc)^{(B_2)}_{n,\ell}) & (n \text{ odd}) \end{cases} \tag{74}$$

and

$$(\delta kc)_{n,\ell} = \begin{cases} (kc)^{(A_1)}_{n,\ell} - (kc)^{(A_2)}_{n,\ell} & (n \text{ even}) \\ (kc)^{(B_1)}_{n,\ell} - (kc)^{(B_2)}_{n,\ell} & (n \text{ odd}). \end{cases} \tag{75}$$

In table 4 and in table 5, the asymptotic approximations (58) and (69) are compared with the exact values for $(\overline{kc})_{n,\ell}$ and $(\delta kc)_{n,\ell}$ defined above. A good agreement is found, and the asymptotic expansions give even better approximations when n and ξ_0 are large. Furthermore it could be possible to improve the asymptotic expansion (58) by looking for higher-order terms in n^{-3}, $n^{-11/3}$, $n^{-13/3}$ etc. As far as it concerns the splitting, its magnitude is correctly described, and it should be noted that it is numerically significant for low frequencies.

3.7. Whispering-gallery modes and physical interpretation

In figures 1 and 2, we display the symmetry breaking between the A_1 and A_2 ($n = 20$) or the B_1 and B_2 ($n = 21$) whispering-gallery modes (the darkest regions correspond to the higher amplitudes).

Finally, in figure 3, we show the behaviour of a given whispering-gallery mode (concentration in a very thick layer close to the boundary of the cavity, and exponential damping outside this layer).

Table 4. Average values and splitting of resonances for $\xi_0 = 1$ and various n.

	$(\overline{kc})_{n,1}$		$(\delta kc)_{n,1}$		$\ln\|(\delta kc)_{n,1}\|$	
n	Exact	Asympt.	Exact	Asympt.	Exact	Asympt.
10	10.546 297	10.546 611	-4.66×10^{-4}	-7.20×10^{-4}	−7.67	−7.24
11	11.363 227	11.363 450	-1.60×10^{-4}	-2.43×10^{-4}	−8.74	−8.32
20	18.562 153	18.562 183	-5.01×10^{-9}	-6.90×10^{-9}	−19.11	−18.79
21	19.350 497	19.350 523	-1.50×10^{-9}	-2.05×10^{-9}	−20.32	−20.00
30	26.383 580	26.383 589	-2.28×10^{-14}	-2.98×10^{-14}	−31.41	−31.14
31	27.159 572	27.159 579	-6.49×10^{-15}	-8.46×10^{-15}	−32.67	−32.40

Table 5. Average values and splitting of resonances for $n = 20$ and various ξ_0.

	$(\overline{kc})_{n,1}$		$(\delta kc)_{n,1}$		$\ln\|(\delta kc)_{n,1}\|$	
ξ_0	Exact	Asympt.	Exact	Asympt.	Exact	Asympt.
0.8	22.463 703	22.463 899	-9.01×10^{-6}	-1.25×10^{-5}	−11.62	−11.29
0.9	20.438 840	20.438 913	-2.28×10^{-7}	-3.14×10^{-7}	−15.29	−14.97
1.0	18.562 153	18.562 183	-5.01×10^{-9}	-6.90×10^{-9}	−19.11	−18.79
1.1	16.837 137	16.837 150	-1.00×10^{-10}	-1.38×10^{-10}	−23.02	−22.70
1.2	15.259 974	15.259 981	-1.89×10^{-12}	-2.60×10^{-12}	−27.00	−26.68
1.3	13.823 017	13.823 020	-3.40×10^{-14}	-4.68×10^{-14}	−31.01	−30.69
1.4	12.516 814	12.516 816	-5.97×10^{-16}	-8.22×10^{-16}	−35.05	−34.74

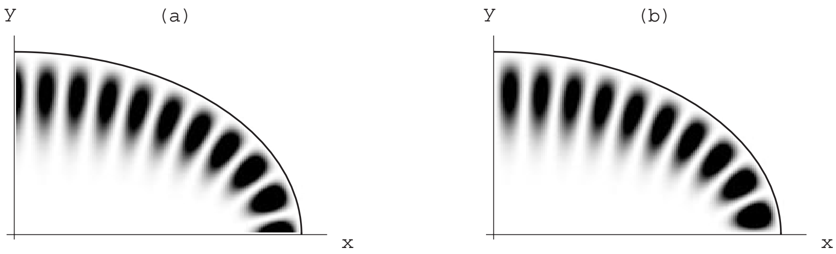

Figure 1. (*a*) $|\Psi_{20}^{(A_1)}|^2$ and (*b*) $|\Psi_{20}^{(A_2)}|^2$ for $\xi_0 = 3/4$ and $0 \leqslant \eta \leqslant \pi/2$.

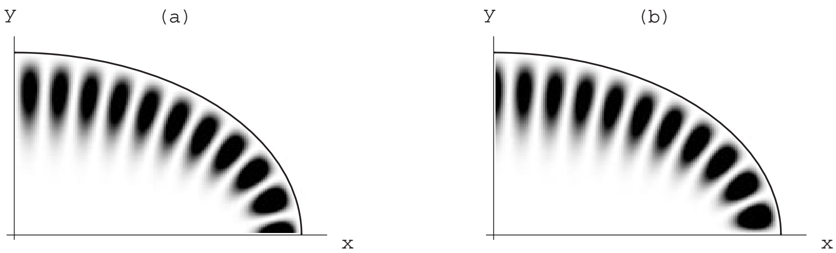

Figure 2. (*a*) $|\Psi_{21}^{(B_1)}|^2$ and (*b*) $|\Psi_{21}^{(B_2)}|^2$ for $\xi_0 = 3/4$ and $0 \leqslant \eta \leqslant \pi/2$.

3.8. The circular cavity as a limiting case of the elliptic one

All the results corresponding to the circular cavity (see section 2) can be recovered from those of the elliptic cavity by taking both the limits $\xi_0 \to +\infty$ and $kc \to 0$, while keeping

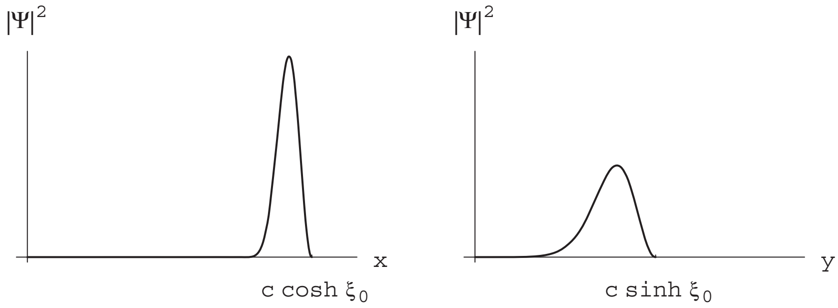

Figure 3. Behaviour of $|\Psi_{20}^{(A_1)}|^2$ along the Ox and Oy axes, for $\xi_0 = 3/4$.

$(kc/2)\exp\xi_0$ constant and equal to the reduced wavenumber ka. Indeed:

- For large ξ, $\cosh\xi \sim \sinh\xi \sim (1/2)\exp\xi$, thus $\rho = (x^2 + y^2)^{1/2}$ is approximately $(c/2)\exp\xi$ and the modified Mathieu equation (21) reduces to the Bessel equation $\rho^2 U''(\rho) + \rho U''(\rho) + (k^2\rho^2 - \nu^2)U(\rho) = 0$, with

$$\nu = kc(b^2 - \tfrac{1}{2})^{1/2}. \tag{76}$$

- Then, for $kc \to 0$, we can make the substitution $\eta \to \varphi$ in the ordinary Mathieu equation (22), which reduces to $V''(\varphi) + \nu^2 V(\varphi) = 0$, and which admits periodic solutions only for $\nu = n \in \mathbb{N}$.

In particular, these considerations permit us to recover the asymptotic expansions (14) for the eigenvalues $\nu_\ell(ka)$ (respectively (15) for the resonances) of the circle from the asymptotic expansions (40) for the eigenvalues $\overline{b}_\ell(kc)$ (respectively (58) for the resonances) of the ellipse.

4. Concluding remarks

In conclusion, we would like to emphasize the two following points:

- The asymptotic expansions constructed in this paper greatly improve the results one can obtain from EBK quantization. Indeed, EBK rules provide only the first two terms of the perturbative series (15) and (58) (see [19] and chapter 4 of [3]). By contrast, EBK quantization rules have a clear physical interpretation that our approach fails to provide.
- The splitting of resonances considered in this paper is a subtle effect that cannot be explained in perturbation theory, since it corresponds to an exponentially small term lying beyond all orders of the perturbative series. However, for reasonably low frequencies, this phenomenon is numerically significant. It could be easily experimentally observed in acoustic or electromagnetic elliptic cavities. Such an experiment has been recently carried out in a microwave annular billiard in order to study chaos-assisted tunnelling [24]. Transmission spectra clearly display the splitting of resonances.

Acknowledgment

We would like to thank Bruce Jensen for useful discussions.

Appendix A. Asymptotic behaviour of the Airy function and Stokes's phenomenon

For $|z| \to \infty$, we can write

$$\mathrm{Ai}(z) \approx \frac{1}{2\sqrt{\pi}} z^{-1/4} \left[\exp\left(-\frac{2}{3} z^{3/2}\right) + \mathrm{i}\sigma(z) \exp\left(\frac{2}{3} z^{3/2}\right) \right] \tag{A.1}$$

and

$$\mathrm{Ai}'(z) \approx -\frac{1}{2\sqrt{\pi}} z^{1/4} \left[\exp\left(-\frac{2}{3} z^{3/2}\right) - \mathrm{i}\sigma(z) \exp\left(\frac{2}{3} z^{3/2}\right) \right]. \tag{A.2}$$

Here, σ is a parameter which permits us to take into account Stokes's phenomenon for the Airy function. It depends on the position of z in the complex plane:

$$\sigma(z) = \begin{cases} -1 & \text{for} \quad \arg(z) \in]-\pi, -2\pi/3[\\ 0 & \text{for} \quad \arg(z) \in]-2\pi/3, 2\pi/3[\\ 1 & \text{for} \quad \arg(z) \in]2\pi/3, \pi[\end{cases} \tag{A.3}$$

while $\sigma(z) = \pm 1/2$ on the Stokes lines $\arg(z) = \pm 2\pi/3$. Here, Stokes's phenomenon is considered as discontinuous. In fact, it has been shown by Berry [23] that the emergence of the small exponential in (A.1) and (A.2) happens rapidly but continuously at the crossing of the Stokes lines.

Appendix B. Dominant and subdominant contributions in the equations (34*a*) and (34*b*)

By considering only the leading contributions in the uniform asymptotic expansions (29) and (30), we can write

$$U'_-(0,b)U_+(\xi,b) = \mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0,b)^{-1/4}\zeta(\xi,b)^{1/4}(b^2-1)^{1/4}(b^2-\cosh^2\xi)^{-1/4} \times \mathrm{Ai}'[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0,b)]\mathrm{Ai}[(kc)^{2/3}\zeta(\xi,b)][1 + \underset{kc\to+\infty}{\mathrm{O}}(1/kc)] \tag{B.1}$$

$$U'_+(0,b)U_-(\xi,b) = (kc)^{2/3}\zeta(0,b)^{-1/4}\zeta(\xi,b)^{1/4}(b^2-1)^{1/4}(b^2-\cosh^2\xi)^{-1/4} \times \mathrm{Ai}'[(kc)^{2/3}\zeta(0,b)]\mathrm{Ai}[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\xi,b)][1 + \underset{kc\to+\infty}{\mathrm{O}}(1/kc)] \tag{B.2}$$

$$U_-(0,b)U_+(\xi,b) = -\zeta(0,b)^{1/4}\zeta(\xi,b)^{1/4}(b^2-1)^{-1/4}(b^2-\cosh^2\xi)^{-1/4} \times \mathrm{Ai}[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(0,b)]\mathrm{Ai}[(kc)^{2/3}\zeta(\xi,b)][1 + \underset{kc\to+\infty}{\mathrm{O}}(1/kc)] \tag{B.3}$$

$$U_+(0,b)U_-(\xi,b) = -\zeta(0,b)^{1/4}\zeta(\xi,b)^{1/4}(b^2-1)^{-1/4}(b^2-\cosh^2\xi)^{-1/4} \times \mathrm{Ai}[(kc)^{2/3}\zeta(0,b)]\mathrm{Ai}[\mathrm{e}^{2\mathrm{i}\pi/3}(kc)^{2/3}\zeta(\xi,b)][1 + \underset{kc\to+\infty}{\mathrm{O}}(1/kc)]. \tag{B.4}$$

By replacing the Airy functions by their asymptotic behaviours for $kc \to +\infty$ (see appendix A), we obtain

$$U'_-(0,b)U_+(\xi,b) = \frac{1}{4\pi}\mathrm{e}^{-\mathrm{i}\pi/6}(kc)^{2/3}(b^2-1)^{1/4}(b^2-\cosh^2\xi)^{-1/4} \times \exp[\tfrac{2}{3}kc(\zeta(0,b)^{3/2} - \zeta(\xi,b)^{3/2})][1 + \underset{kc\to+\infty}{\mathrm{O}}(1/kc)] \tag{B.5}$$

$$U'_+(0,b)U_-(\xi,b) = -\frac{1}{4\pi}\mathrm{e}^{-\mathrm{i}\pi/6}(kc)^{2/3}(b^2-1)^{1/4}(b^2-\cosh^2\xi)^{-1/4} \times \exp[-\tfrac{2}{3}kc(\zeta(0,b)^{3/2} - \zeta(\xi,b)^{3/2})][1 + \underset{kc\to+\infty}{\mathrm{O}}(1/kc)] \tag{B.6}$$

$$U_-(0,b)U_+(\xi,b) = -\frac{1}{4\pi}\mathrm{e}^{-\mathrm{i}\pi/6}(kc)^{-1/3}(b^2-1)^{-1/4}(b^2-\cosh^2\xi)^{-1/4}$$

$$\times \exp[\tfrac{2}{3}kc(\zeta(0,b)^{3/2} - \zeta(\xi,b)^{3/2})][1 + \underset{kc\to+\infty}{\mathrm{O}}(1/kc)] \tag{B.7}$$

$$U_+(0,b)U_-(\xi,b) = -\frac{1}{4\pi}\mathrm{e}^{-\mathrm{i}\pi/6}(kc)^{-1/3}(b^2-1)^{-1/4}(b^2-\cosh^2\xi)^{-1/4}$$
$$\times \exp[-\tfrac{2}{3}kc(\zeta(0,b)^{3/2} - \zeta(\xi,b)^{3/2})][1 + \underset{kc\to+\infty}{\mathrm{O}}(1/kc)]. \tag{B.8}$$

We assume $b \in [1,+\infty[$. As a consequence, $\mathrm{arccosh} b \in \mathbb{R}^+$ and the real part of $\zeta(0,b)^{3/2} - \zeta(\xi,b)^{3/2}$ is positive. Thus, $U'_+(0,b)U_-(\xi,b)$ is exponentially small with respect to $U'_-(0,b)U_+(\xi,b)$ while $U_+(0,b)U_-(\xi,b)$ is exponentially small with respect to $U_-(0,b)U_+(\xi,b)$.

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